Europ. J. Combinatorics (2002) **23**, 15–22
doi:10.1006/eujc.2000.0470
Available online at http://www.idealibrary.com on IDEAL®

On a Problem Concerning the Weight Functions

GIAMPIERO CHIASELOTTI[†]

Let X be a finite set with n elements. A function $f : X \longrightarrow \mathbf{R}$ such that $\sum_{x\in X} f(x) \geq 0$ is called a n-weight function. In 1988 Manickam and Singhi conjectured that, if d is a positive integer and f is a n-weight function with $n \geq 4d$ there exist at least $\binom{n-1}{d-1}$ subsets Y of X with $|Y| = d$ for which $\sum_{y\in Y} f(y) \geq 0$. In this paper we study this conjecture and we show that it is true if f is a n-weight function and $|\{x \in X : f(x) \geq 0\}| \leq d \leq \frac{n}{2}$.

1. INTRODUCTION

Let $n \in \mathbf{N}$ and let I_n be the set $\{1, 2, \ldots, n\}$.

A function $f : I_n \to \mathbf{R}$ is called a *n-weight function* if

$$\sum_{x\in I_n} f(x) \geq 0.$$

We introduce some useful notation that we shall use in the remainder of this paper.

$W_n(\mathbf{R})$ will denote the set of all n-weight functions and if $f \in W_n(\mathbf{R})$ we define

$$f^+ = |\{x \in I_n : f(x) \geq 0\}|.$$

If d is an integer with $1 \leq d \leq n$ and Y is a subset of I_n having d elements such that

$$\sum_{y\in Y} f(y) \geq 0,$$

we call Y a (d^+, n)-*subset* for f.

If $f \in W_n(\mathbf{R})$ and d, r are two fixed integers with $1 \leq d, r \leq n$, we shall denote by $\phi(f, d)$ the number of distinct (d^+, n)-subsets for f,

i.e.,

$$\phi(f, d) = \left|\left\{Y \subseteq I_n : |Y| = d \quad \text{and} \quad \sum_{y\in Y} f(y) \geq 0\right\}\right|;$$

moreover we shall set

$$\psi(n, d) = \min\{\phi(f, d) : f \in W_n(\mathbf{R})\}$$

and

$$\gamma(n, d, r) = \min\{\phi(f, d) : f \in W_n(\mathbf{R}),\ f^+ = r\}.$$

For any subset Y of I_n, we set

$$S_f(Y) = \sum_{y\in Y} f(y);$$

in particular, if Y is a singleton $\{y\}$, it is clear that $S_f(Y) = f(y)$.

[†]Address for correspondence: Giampiero Chiaselotti, Via S. Francesco 10, 87018 San Marco Argentano (Cosenza), Italy. *E-mail*: *g.chiaselotti@virgilio.it*

Observe that, for every finite set X with n elements, a function $f : X \longrightarrow \mathbf{R}$ such that $\sum_{x\in X} f(x) \geq 0$ can be considered a n-weight function, since X can be identified by I_n using a suitable bijection.

Finally, we call each $x \in I_n$ such that $f(x) \geq 0$ a *non-negative element* for f and each $x \in I_n$ with $f(x) < 0$ a *negative element* for f.

In [1] Manickam and Singhi stated the following conjecture.

CONJECTURE MS. *If d is a positive integer and f is a n-weight function with $n \geq 4d$, then $\phi(f, d) \geq \binom{n-1}{d-1}$.*

Let us now reformulate Conjecture MS in a slightly different (but equivalent) way.

Let $n \geq 1$ be fixed and $1 \leq d \leq n$; we consider the following n-weight function:

$$g : \begin{array}{ccccc} x_1 & x_2 & \cdots & x_{n-1} & x_n \\ \downarrow & \downarrow & \cdots & \downarrow & \downarrow \\ n-1 & -1 & \cdots & -1 & -1 \end{array} ;$$

then

$$\phi(g, d) = \binom{n-1}{d-1}.$$

This implies that

$$\psi(n, d) \leq \binom{n-1}{d-1}$$

for each integer d with $1 \leq d \leq n$.

Therefore we can reformulate Conjecture MS as follows.

If $n \geq 4d$ then

$$\psi(n, d) = \binom{n-1}{d-1}.$$

In [2] Bier and Manickam proved that Conjecture MS is not true for all pairs (n, d). They gave examples of pairs (n, d) such that $\psi(n, d) < \binom{n-1}{d-1}$, e.g. for $n = 2d + 1$ with $d \geq 2$ and for $n = 3d + 1$ with $d \geq 3$.

In this paper we study Conjecture MS and we show that in an infinite number of cases it is true also if $n \geq 2d$.

First we show that

$$\psi(n, d) \geq 1,$$

for each integer d with $1 \leq d \leq n$.

Next, we prove that

$$\gamma(n, d, r) = \begin{cases} \binom{n-1}{d-1} & \text{if } r \leq d \leq \frac{n}{2} \\ \binom{n-r}{d-r} & \text{if } r \leq d < n \quad \text{and} \quad r < \frac{n}{n-d} \\ \binom{r}{d} & \text{if } d < r < n \quad \text{and} \quad r > \frac{d-1}{d} n \\ \binom{n-1}{d-1} & \text{if } r = 1. \end{cases}$$

In the remaining cases

(i) $r \geq \frac{n}{n-d}$ with $\frac{n}{2} < d < n$ and $2 \leq r \leq d$,

(ii) $r \leq \frac{d-1}{d} n$ with $d < r < n$, the problem to estimate the numbers $\gamma(n,d,r)$ is open. It is clear that a complete knowledge of the numbers $\psi(n,d)$ implies the resolution of Conjecture MS. Moreover, since

$$\psi(n,d) = \min\{\gamma(n,d,r) : 1 \le r \le n\}, \tag{1}$$

we are interested in estimating the numbers $\gamma(n,d,r)$. In what follows we shall denote by $x_1,\dots,x_n$ the elements of I_n; moreover, if f is a n-weight function with $f^+ = r$, after a suitable reordering of the indexes we can assume that $x_1,\dots,x_r$ are the non-negative elements for f and $x_{r+1},\dots,x_n$ those negative.

2. The Results

PROPOSITION 1. *Let n be a positive integer and $f \in W_n(\mathbf{R})$. If d is an integer such that $1 \le d \le n$, there exists a (d^+,n)-subset for f. Hence $\psi(n,d) \ge 1$.*

PROOF. We use induction. The case $d = n$ is obviously true. Suppose the proposition holds for $d = m \ge 2$; we will show that it is true for $d = m-1$. Suppose, by way of contradiction, that there does not exist any subset Y of I_n of $m-1$ elements with $S_f(Y) \ge 0$.

By hypothesis there exists a subset Z of m elements with $S_f(Z) \ge 0$.

Let $y \in Z$ and $Y = Z \setminus \{y\}$. Since

$$S_f(Z) = S_f(Y) + f(y) \tag{2}$$

and $S_f(Y) < 0$, by (2) we have $f(y) > 0$.

Since y was chosen arbitrary we have $f(y) > 0$ for all $y \in Z$. In particular, any subset $T \subseteq Z$ with $m-1$ elements is such that $S_f(T) > 0$: a contradiction. Hence, if d is an integer with $1 \le d \le n$, there always exists (at least) a subset Y of I_n of d elements with $S_f(Y) \ge 0$. $\square$

REMARK 1. Let d, r be two integers with $1 \le d, r \le n$; then

$$\gamma(n,d,r) \ge \binom{r}{d} \quad \text{if} \quad 1 \le d < r \tag{i}$$

and

$$\gamma(n,d,r) \ge \binom{n-r}{d-r} \quad \text{if} \quad r \le d \le n. \tag{ii}$$

Indeed, let f be a n-weight function such that $f^+ = r$. Then, if $1 \le d \le r$, each subset of $\{x_1,\dots,x_r\}$ with d elements is a (d^+,n)-subset for f; since there exist $\binom{r}{d}$ of such subsets, we have $\phi(f,d) \ge \binom{r}{d}$. Hence (i) holds. If $d > r$, each subset Y of I_n of the form

$$Y = \{x_1,\dots,x_r,x_{i_{r+1}},\dots,x_{i_d}\}, \tag{3}$$

where $i_{r+1},\dots,i_d \in \{r+1,\dots,n\}$, is such that

$$\begin{aligned} S_f(Y) &= f(x_1) + \cdots + f(x_r) + f(x_{i_{r+1}}) + \cdots + f(x_{i_d}) \\ &\ge f(x_1) + \cdots + f(x_r) + f(x_{r+1}) + \cdots + f(x_n) \\ &= S_f(I_n) \ge 0. \end{aligned}$$

Hence every subset Y as in (3) is a (d^+,n)-subset for f. Choosing in all possible ways the indexes $i_{r+1},\dots,i_d$ in $\{r+1,\dots,n\}$, we obtain $\binom{n-r}{d-r}$ (d^+,n)-subsets as in (3); this proves that $\phi(f,d) \ge \binom{n-r}{d-r}$. Hence (ii) holds.

Let us observe that the inequalities (i) and (ii) in Remark 1 also provide a different proof of Proposition 1 by virtue of (1).

PROPOSITION 2. *Let n, d, r be positive integers with $r \leq d \leq \frac{n}{2}$, then*

$$\gamma(n, d, r) = \binom{n-1}{d-1}.$$

PROOF. Let f be a n-weight function having non-negative elements $x_1, \ldots, x_r$ and negative elements $x_{r+1}, \ldots, x_n$. By Remark 1(ii) we know that $\gamma(n, d, r) \geq \binom{n-r}{d-r}$, taking all (d^+, n)-subsets of I_n as in (10).

Without loss of generality we may assume that $f(x_1) \geq f(x_2) \geq \cdots \geq f(x_r) \geq 0$.

We fix the element x_1 and proceed as follows.

Let $P = \{x_2, \ldots, x_r\}$, $N = \{x_{r+1}, \ldots, x_n\}$ and choose an arbitrary subset A of P with $r-2$ elements and an arbitrary subset B of N with $d - r + 1$ elements. The subset $A \cup B \cup \{x_1\}$ has d elements and $(P \setminus A) \cup (N \setminus B)$ has $n - d$ elements.

If

$$S_f(A \cup B \cup \{x_1\}) < 0,$$

then (since $f \in W_n(\mathbf{R})$) we have

$$S_f((P \setminus A) \cup (N \setminus B)) \geq 0. \tag{4}$$

Consider now the set $(P \setminus A) \cup (N \setminus B)$.

$P \setminus A$ is a singleton $\{x\}$ $(x \neq x_1)$ and $N \setminus B$ has $n - d - 1$ negative elements for f.

Since $d \leq \frac{n}{2}$, we have $n - d - 1 \geq d - 1$; therefore, if (4) holds, choosing arbitrarily $d - 1$ elements in $N \setminus B$, say $y_1, \ldots, y_{d-1}$, it follows that the set

$$Y = \{x, y_1, \ldots, y_{d-1}\} \tag{5}$$

is a (d^+, n)-subset for f.

This proves that at least one of the following holds:

(i) $A \cup B \cup \{x_1\}$ is a (d^+, n)-subset for f,

(ii) there exists (at least) a subset $Y \subset (P \setminus A) \cup (N \setminus B)$ (of type (5)) which is a (d^+, n)-subset for f.

Therefore for any choice of $A \subseteq P$, $B \subseteq N$, with $|A| = r - 2$ and $|B| = d - r + 1$, we can choose a (d^+, n)-subset for f, say $D_{A,B}$, of type (i) or (ii). We observe now that the correspondence

$$(A, B) \longrightarrow D_{A,B}$$

is injective since $x_1 \notin P \cup N$; hence, proceeding in this way, we obtain

$$\binom{r-1}{r-2}\binom{n-r}{d-r+1}$$

distinct (d^+, n)-subsets for f.

We apply the previous method again as follows.

We take a set A' containing $r - 3$ arbitrary elements of P and an arbitrary subset B' with $d - r + 2$ elements. Using the previous notation we still obtain a situation analogous to (i) and (ii), where $P \setminus A'$ is a set with two non-negative elements $x, \hat{x}$ and $N \setminus B'$ has $n - d - 2$

negative elements. Since $n-d-2 \geq d-2$, choosing arbitrarily $d-2$ elements in $N \setminus B'$, say $z_1, \ldots, z_{d-2}$, we obtain a set

$$Y = \{x, \hat{x}, z_1, \ldots, z_{d-2}\} \tag{6}$$

which is a (d^+, n)-subset for f.

It follows that at least one of the following analogues of (i) and (ii) holds:

(i′) $A' \cup B' \cup \{x_1\}$ is a (d^+, n)-subset for f,
(ii′) there exists (at least) a subset $Y \subset (P \setminus A') \cup (N \setminus B')$ (of type (6)) which is a (d^+, n)-subset for f.

As before, since $x_1 \notin P \cup N$, we obtain

$$\binom{r-1}{r-3}\binom{n-r}{d-r+2}$$

distinct (d^+, n)-subsets for f and it is immediate to verify that no set of type (i′) or (ii′) can also be a set of type (i) or (ii).

This implies that the new (d^+, n)-subsets are all distinct from the preceding ones.

Proceeding in this way, at the kth step we obtain

$$\binom{r-1}{r-k-1}\binom{n-r}{d-r+k}$$

new (d^+, n)-subsets, all distinct from the preceding ones.

Hence we find at least

$$\begin{aligned}\binom{n-r}{d-r} + \sum_{k\geq 1}\binom{r-1}{r-k-1}\binom{n-r}{d-r+k} &= \sum_{k\geq 0}\binom{r-1}{r-k-1}\binom{n-r}{d-r+k} \\ &= \sum_{\substack{s\geq 0\\ s+t=d-1}}\binom{r-1}{s}\binom{n-r}{t} = \binom{n-1}{d-1}\end{aligned}$$

(d^+, n)-subsets for f.

This proves that $\phi(f, d) \geq \binom{n-1}{d-1}$ if $f \in W_n(\mathbf{R})$, $f^+ = r$ and $r \leq d \leq \frac{n}{2}$, i.e., $\gamma(n, d, r) \geq \binom{n-1}{d-1}$. On the other hand, consider the following n-weight function:

$$g: \begin{array}{cccccccc} x_1 & x_2 & \ldots & x_r & x_{r+1} & \ldots & x_n \\ \downarrow & \downarrow & \ldots & \downarrow & \downarrow & \ldots & \downarrow \\ 1 & 0 & & 0 & -\frac{1}{n-r} & & -\frac{1}{n-r} \end{array}.$$

It is obvious that, if $r \leq d$,

$$\begin{aligned}\phi(g, d) &= \binom{n-r}{d-r} + \binom{r-1}{r-2}\binom{n-r}{d-r+1} + \cdots + \binom{n-r}{d-1} \\ &= \binom{n-1}{d-1};\end{aligned}$$

therefore, since $g^+ = r$, we have

$$\binom{n-1}{d-1} = \phi(g, d) \geq \gamma(n, d, r),$$

i.e.,

$$\gamma(n,d,r) = \binom{n-1}{d-1}. \qquad \square$$

REMARK 2. We note that if $r = 1$ the condition $d \leq \frac{n}{2}$ is unnecessary in Proposition 2.

PROPOSITION 3. *Let n, d, r be positive integers with $r \leq d < n$ and $r < \frac{n}{n-d}$; then*

$$\gamma(n,d,r) = \binom{n-r}{d-r}.$$

PROOF. By Remark 1(ii) we know that $\gamma(n,d,r) \geq \binom{n-r}{d-r}$.
To obtain an upper bound for $\gamma(n,d,r)$, we take the function

$$g : \begin{array}{cccccc} x_1 & \dots & x_r & x_{r+1} & \dots & x_n \\ \downarrow & & \downarrow & \downarrow & & \downarrow \\ \frac{1}{r} & \dots & \frac{1}{r} & -\frac{1}{n-r} & \dots & -\frac{1}{n-r} \end{array}.$$

Then g is a n-weight function with $g^+ = r$ and the only (d^+, n)-subsets for g are those which contain all non-negative elements $x_1, \dots, x_r$ and arbitrary $d - r$ negative elements. Indeed since no subset of the form

$$\{x_{i_1}, \dots, x_{i_k}, x_{j_1}, \dots, x_{j_{d-k}}\},$$

where $i_1, \dots, i_k \in \{1, \dots, r\}$, $j_1, \dots, j_{d-k} \in \{r+1, \dots, r\}$, $1 \leq k \leq r-1$, can be a (d^+, n)-subsets for g. In fact, the condition

$$\begin{aligned} &g(x_{i_1}) + \dots + g(x_{i_k}) + g(x_{j_1}) + \dots + g(x_{j_{d-k}}) \\ &= k\frac{1}{r} + (d-k)\left(-\frac{1}{n-r}\right) < 0, \end{aligned}$$

with $1 \leq k \leq r-1$, is equivalent to

$$d > \frac{k}{r}n, \qquad \text{with} \qquad 1 \leq k \leq r-1,$$

which is satisfied for each $k \in \{1, \dots, r-1\}$, since

$$d > \frac{r-1}{r}n$$

by hypothesis. $\square$

PROPOSITION 4. *Let n, d, r be positive integers with*

$$r > \frac{d-1}{d}n \qquad \text{and} \qquad d < r < n;$$

then

$$\gamma(n,d,r) = \binom{r}{d}.$$

PROOF. By Remark 1(i) we have $\gamma(n,d,r) \geq \binom{r}{d}$. To obtain an upper bound for $\gamma(n,d,r)$, consider the function

$$g: \begin{array}{ccccccc} x_1 & \dots & x_r & x_{r+1} & \dots & x_n \\ \downarrow & & \downarrow & \downarrow & & \downarrow \\ \frac{1}{r} & \dots & \frac{1}{r} & -\frac{1}{n-r} & \dots & -\frac{1}{n-r} \end{array}.$$

We observe that the only (d^+,n)-subsets for g are all the subsets with d elements of $\{x_1,\dots,x_r\}$, since no subset of the form

$$\{x_{i_1},\dots,x_{i_{d-k}},x_{j_1}\dots,x_{j_k}\},$$

where $i_1,\dots,i_{d-k} \in \{1,\dots,r\}$, $j_1,\dots,j_k \in \{r+1,\dots,n\}$ and $1 \leq k \leq d-1$, can be a (d^+,n)-subsets for g. In fact, the condition

$$\begin{aligned} &g(x_{i_1})+\dots+g(x_{i_{d-k}})+g(x_{j_1})+\dots+g(x_{j_k}) \\ &= (d-k)\frac{1}{r}+k\left(-\frac{1}{n-r}\right)<0, \end{aligned}$$

is equivalent to $d < k\frac{n}{n-r}$, which is satisfied for each $k \in \{1,\dots,d-1\}$ since $d < \frac{n}{n-r}$ by hypothesis. □

REMARK 3. If $r > d$, the n-weight function

$$g: \begin{array}{ccccccc} x_1 & x_2 & \dots & x_r & x_{r+1} & \dots & x_n \\ \downarrow & \downarrow & \dots & \downarrow & \downarrow & \dots & \downarrow \\ 1 & 0 & & 0 & -\frac{1}{n-r} & & -\frac{1}{n-r} \end{array}$$

has

$$\begin{aligned} \phi(g,d) &= \binom{r}{d}+\binom{r-1}{d-2}\binom{n-r}{1}+\dots+\binom{r-1}{0}\binom{n-r}{d-1} \\ &= \binom{r-1}{d}+\sum_{k\geq 0}\binom{r-1}{d-k-1}\binom{n-r}{k} \\ &= \binom{r-1}{d}+\binom{n-1}{d-1}. \end{aligned}$$

This implies that

$$\gamma(n,d,r) \leq \binom{r-1}{d}+\binom{n-1}{d-1}$$

if $r > d$. We conjecture the following result.

CONJECTURE 1. *If $d < r \leq \frac{d-1}{d}n$ and $d \leq \frac{n}{2}$ then*

$$\gamma(n,d,r) = \binom{r-1}{d}+\binom{n-1}{d-1}.$$

Actually this conjecture has been proved by J. van Bon when $d = 2$ (private communication).

Finally let us note how the previous result and the truth of Conjecture 1 are related to Conjecture MS.

PROPOSITION 5. *The truth of Conjecture* 1 *implies Conjecture MS.*

PROOF. We can assume that $n \geq 4d$; therefore $d \leq \frac{n}{4} \leq \frac{n}{2}$.

If $r \leq d$, by Proposition 2 we have $\gamma(n, d, r) = \binom{n-1}{d-1}$.

If $d < r \leq \frac{d-1}{d}n$, by Conjecture 1 it follows that $\gamma(n, d, r) = \binom{r-1}{d} + \binom{n-1}{d-1}$.

If $d < r < n$ and $r > \frac{d-1}{d}n$, by Proposition 4 we have

$$\gamma(n, d, r) = \binom{r}{d} \geq \binom{\frac{d-1}{d}n}{d}. \tag{7}$$

Now, if d is an integer ≥ 3 and x is a real number ≥ 4, by elementary calculus it results that

$$\binom{(d-1)x}{d} > \binom{dx-1}{d-1}. \tag{8}$$

Hence, if in (7) we take $d \geq 3$ and $x = \frac{n}{d}$, by (8) we obtain $\gamma(n, d, r) > \binom{n-1}{d-1}$.

If $2 = d < r < n$ and $r > \frac{d-1}{d}n = \frac{n}{2}$, we have

$$\gamma(n, d, r) = \binom{r}{2} \geq \begin{cases} \binom{\frac{n+2}{2}}{2} & \text{if } n \text{ is even} \\ \binom{\frac{n+1}{2}}{2} & \text{if } n \text{ is odd} \end{cases} = \begin{cases} \frac{n(n+2)}{8} & \text{if } n \text{ is even} \\ \frac{n^2-1}{8} & \text{if } n \text{ is odd} \end{cases} \geq \binom{n-1}{2-1} = n-1$$

when $n \geq 6$ (and by our hypothesis $n \geq 4d = 8$).

Finally if $1 = d < r < n$, $\binom{r}{d} = r$ and $\binom{n-1}{d-1} = 1$.

Therefore if Conjecture 1 is true and if $n \geq 4d$, these results show that $\gamma(n, d, r) \geq \binom{n-1}{d-1}$ for each integer r with $1 \leq r \leq n$. By (1) we obtain then $\psi(n, d) \geq \binom{n-1}{d-1}$.

Hence also Conjecture MS is true. □

ACKNOWLEDGEMENTS

The author would like to thank R Dvornicich for suggesting the problem and for his comments, J van Bon for many useful remarks and the referee for suggesting Proposition 5.

REFERENCES

1. N. Manickam and N. M. Singhi, First distribution invariants and EKR theorems, *J. Comb. Theory, Series A*, **48** (1988), 91–103.
2. T. Bier and N. Manickam, The first distribution invariant of the Johnson-scheme, *SEAMS Bull. Math.*, **11** (1987), 61–68.

Received 22 February 2000 and accepted 7 November 2000

GIAMPIERO CHIASELOTTI
Dipartimento di Matematica,
Universitá della Calabria,
87036 Arcavacata di Rende (Cosenza),
Italy
E-mail: chiaselotti@unical.it